# Robust and Efficient Computation of Eigenvectors in a Generalized Spectral Method for Constrained Clustering


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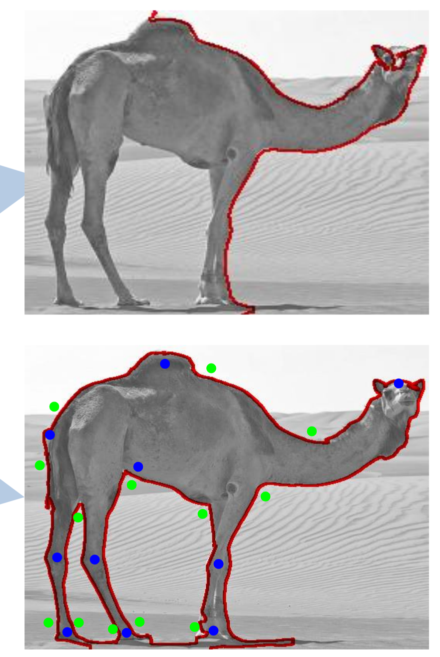
## Motivation and Objectives

- **Image Segmentation**

Unconstrained:
Normalized-Cut (NCut) and
Spectral-Clustering(SC)

Constrained:
Must-Link (ML) and
Cannot-Link (CL) constraints



- **Objectives**

Mathematically rigorous and computationally effective treatments of FAST-GE model for constrained clustering

## Ncut and FAST-GE

| Model | NCut [Shi and Malik, PAMI 2000] | FAST-GE [Cucuringu et al, AISTATS 2016] |
|---|---|---|
| Laplacian | L | $L_G$, $L_H$ with embedded ML and CL constraints |
| Objective Function | $\frac{cut(A)}{vol(A)}+\frac{cut(\bar{A})}{vol(\bar{A})}$ | $\frac{cut_G(A)}{cut_H(A)}$ |
| Rayleigh Quotient Opt. | $\min_x \frac{x^TLx}{x^TDx}$ s.t. $x^TD1=0$ <br> (1) x is binary <br> (2) $L \geq 0$, degree matrix $D > 0$ | $\min_{x^TL_Hx>0} \frac{x^TL_Gx}{x^TL_Hx}$ <br> (1) x is binary <br> (2) $L_G \geq 0$, $L_H \geq 0$ <br> (3) $L_G - \lambda L_H$ is singular |
| Relaxation | $\min_{x\in R^n} \frac{x^TLx}{x^TDx}$ s.t. $x^TD1=0$ | $\inf_{x\in R^n, x^TL_Hx>0} \frac{x^TL_Gx}{x^TL_Hx}$ |
| Variational Principle | Courant-Fischer | **?** |
| Eigenvalue Problem | $Lx=\lambda Dx$ | **?** |

## Variational Principles

- **Courant-Fischer variational principle**

For a symmetric matrix $L\in R^{n\times n}$, $\lambda_i = \max_{S\subseteq R^n, \dim(S)=n-i+1} \min_{x\in S, x\neq 0} \frac{x^TLx}{x^Tx}$

- **An extended Courant-Fischer variational principle**

For symmetric matrices $L_G, L_H \in R^{n\times n}, L_G, L_H \geq 0$ and r = rank($L_H$),

(1) $L_G - \lambda L_H$ has r finite eigenvalues $0 \leq \lambda_1 \leq \dots \leq \lambda r$

(2) $\lambda_i = \max_{S\subseteq R^n, \dim(S)=n-i+1} \min_{x\in S, x^TL_Hx>0} \frac{x^TL_Gx}{x^TL_Hx}$

- **The extended variational principle implies that**

(1) $\inf_{x\in R^n, x^TL_Hx>0} \frac{x^TL_Gx}{x^TL_Hx} = \min_{x\in R^n, x^TL_Hx>0} \frac{x^TL_Gx}{x^TL_Hx} = \lambda_1$ (2) minimizer $x_1$: $L_Gx_1 = \lambda_1 L_Hx_1$

## Eigenvalue Problems

- **The eigenproblem:** $L_Gx = \lambda L_Hx$, where $L_G$, $L_H \geq 0$ and $L_G - \lambda L_H$ is singular.

- **Regularization**

Let

$$K = -L_H \quad \text{and} \quad M = L_G + \mu L_H + ZSZ^T$$

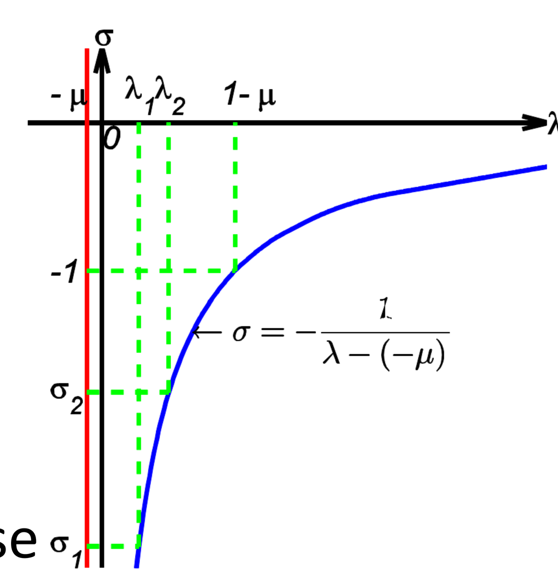
where Z = null($L_G$)∩null($L_H$), S is SPD and μ > 0 is a scalar. Then:

(1) M > 0

(2) The eigenvalues of K – λM are $\sigma_1 \leq \dots \leq \sigma_r < \sigma_{r+1} = \dots = \sigma_n = 0$,
where $\sigma_i = -1/(\lambda_i + \mu)$ for i = 1, 2, … , r

- **The generalized symmetric definite eigenvalue problem**

$$Kx = \sigma Mx$$



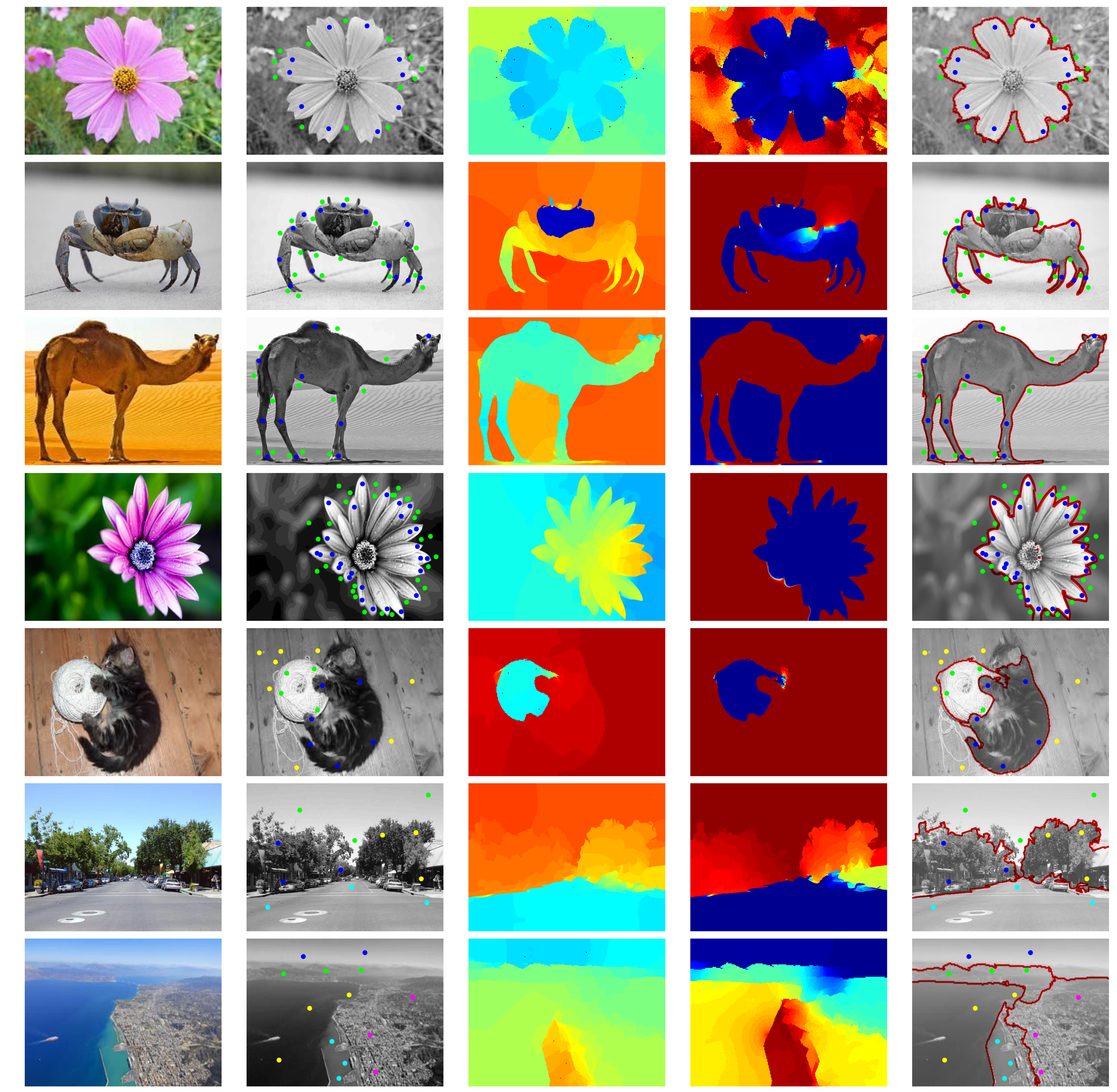

(1) High-quality solvers (Lanczos and LOBPCG)

(2) Spectral enhancement without explicit inverse

(3) "Natural" preconditioner $T \approx M^{-1}$

(4) Efficient sparse-plus-low-rank matrix-vector multiplications

## Constrained Image Segmentation



Image, constraints, eigvec, renormalized eigvec, const. seg.

| Image | Pixels n | Clusters | Constraints | t_eig(sec) | t_total(sec) |
|---|---|---|---|---|---|
| Flower | 30,000 | 2 | 23 | 5.88 | 7.14 |
| Crab | 143,000 | 2 | 32 | 55.71 | 62.14 |
| Camel | 249,057 | 2 | 23 | 144.94 | 159.67 |
| Daisy | 1,024,000 | 2 | 58 | 1518.88 | 1677.51 |
| Cat | 50,325 | 3 | 18 | 12.45 | 15.16 |
| Davis | 235,200 | 4 | 12 | 77.9 | 89.75 |
| Patras | 44,589 | 5 | 14 | 11.81 | 13.72 |

## Concluding Remarks

1. Provided mathematical foundation and numerical treatment for the FAST-GE model
2. Eigensolver is still the computational bottleneck
3. Future studies: (1) exploiting structures of Laplacians for HPC, (2) other apps such as generalized linear discriminant analysis and multi-surface classification
4. https://github.com/aistats2017239/fastge2