Working Notes on the Fix-Heiberger reduction algorithm
for solving the ill-conditioned generalized symmetric eigenvalue problem
prepared by C. Jiang and Z. Bai, April 21, 2015[1]

1. Introduction. The generalized symmetric eigenvalue problem (GSEP) is of the form

$$Ax = \lambda Bx, \tag{1}$$

where A and B are $n \times n$ real symmetric matrices, and B is positive definite. LAPACK routine DSYSV is a standard solver for the GSEP. In this notes, we describe a LAPACK-style routine for solving the GSEP, where B is positive semi-definite with respect to a prescribed threshold ε, where $0 < \varepsilon \ll 1$. In this case, the problem is called an *ill-conditioned* GSEP [2, 3].

With respect to a prescribed threshold ε, LAPACK-style routine DSYGVIC determines (a) $A - \lambda B$ is regular and has k ε-stable eigenvalues, where $0 \le k \le n$; or (b) $A - \lambda B$ is singular, namely $\det(A - \lambda B) \equiv 0$ for any λ. It can be shown that the pencil $A - \lambda B$ is singular if and only if $\mathcal{N}(A) \cap \mathcal{N}(B) \neq \{0\}$[1], where $\mathcal{N}(Z)$ is the column null space of the matrix Z.

2. New LAPACK-style routine DSYGVIC.

The new routine DSYGVIC has the following calling sequence:

```
DSYGVIC( ITYPE, JOBZ, UPLO, N, A, LDA, B, LDB, ETOL, K, W, &
         WORK, LDWORK, WORK2, LWORK, IWORK, INFO )
```

Input to DSYGVIV:

ITYPE: Specifies the problem type to be solved: ITYPE $= 1$ only.

JOBZ: = 'V': Compute eigenvalues and eigenvectors.

UPLO: = 'U': Upper triangles of A and B are stored;
= 'L': Lower triangles of A and B are stored.

N: The order of the matrices A and B. N > 0.

A, LDA: The matrix A and the leading dimension of the array A. LDA $\ge \max(1, \texttt{N})$.

B, LDB: The matrix B and the leading dimension of the array B. LDB $\ge \max(1, \texttt{N})$.

ETOL: The parameter used to drop small eigenvalues of B.

WORK, LDWORK: The workspace matrix and the leading dimension of the array WORK. LDWORK $\ge \max(1, \texttt{N})$.

WORK2, LWORK: The workspace array and its dimension. LWORK $\ge \max(1, 3 * \texttt{N} + 1)$. For optimal performance LWORK $\ge 2 * \texttt{N} + (\texttt{N} + 1) * \texttt{NB}$ where NB is the optimal block size.
If LWORK $= -1$, then a workspace query is assumed; the routine only calculates the optimal size of the WORK2 array, returns this value as the first entry of the WORK2 array.

IWORK: The integer workspace array, dimension N.

Output from DSYGVIC:

A: Contains the eigenvectors matrix X in the first K(1) columns of A.

[1]Chengming Jiang and Zhaojun Bai, Department of Computer Science, University of California, Davis, CA 95616, cmjiang@ucdavis.edu and zbai@ucdavis.edu.

B: Contains the transformation matrix $Q_1R_1Q_2Q_3$, depending on the exit stage.

K: K(1) indicates the number of finite eigenvalues if INFO $= 0$; K(2) indicates the case number.

W: If K(1) > 0, W stores the K(1) eigenvalues.

INFO: $= 0$ then successful exit. If INFO $= -i$, the i-th argument had an illegal value.

3. Algorithm. LAPACK-style routine DSYGVIC is based on an algorithm first presented by Fix and Heiberger [2], also see [4, section 15.5]. The algorithm consists of the following three phases:

- **Phase 1.**

1. Compute the eigenvalue decomposition of B:

$$B^{(0)} = Q_1^T B Q_1 = D = \begin{matrix} & n_1 & n_2 \\ n_1 & D^{(0)} & \\ n_2 & & E^{(0)} \end{matrix}$$

where the diagonal entries of $D^{(0)} = \mathrm{diag}(d_{ii}^{(0)})$ are sorted in descending order and the diagonal elements of $E^{(0)}$ are smaller than $\varepsilon \cdot d_{11}^{(0)}$.

2. *Early Exit*: If $n_1 = 0$, then B is a "zero" matrix with respect to ε and
   (a) if $\det(A) = 0$, then $A - \lambda B$ is singular. Program exits with output parameter `(K(1),K(2)) = (-1, 1)`.
   (b) if $\det(A) \neq 0$, $A - \lambda B$ is regular, but no finite eigenvalue. Program exits with output parameter `(K(1),K(2)) = (0, 1)`.

3. Update A:

$$A^{(0)} = Q_1^T A Q_1$$

4. Set $E^{(0)} = 0$, and update $A^{(0)}$ and $B^{(0)}$:

$$A^{(1)} = R_1^T A^{(0)} R_1 = \begin{matrix} & n_1 & n_2 \\ n_1 & A_{11}^{(1)} & A_{12}^{(1)} \\ n_2 & A_{12}^{(1)T} & A_{22}^{(1)} \end{matrix} \quad \text{and} \quad B^{(1)} = R_1^T B^{(0)} R_1 = \begin{matrix} & n_1 & n_2 \\ n_1 & I & \\ n_2 & & 0 \end{matrix}$$

where

$$R_1 = \begin{matrix} & n_1 & n_2 \\ n_1 & (D^{(0)})^{-1/2} & \\ n_2 & & I \end{matrix}$$

5. *Early Exit:* If $n_2 = 0$, then B is a ε-well-conditioned matrix and $B^{(1)} = I$. There are n ε-stable eigenvalues of the GSEP (1), which are the eigenvalues of $A^{(1)}$:

$$A^{(1)} U = U \Lambda. \tag{2}$$

The n eigenpairs of the GSEP (1) are $(\Lambda, X = Q_1 R_1 U)$. Program exits with output parameter (K(1),K(2)) = (n, 1).

- **Phase 2.**

1. Compute the eigenvalue decomposition of the (2,2) block $A_{22}^{(1)}$ of $A^{(1)}$:

$$A_{22}^{(2)} = Q_{22}^{(2)T} A_{22}^{(1)} Q_{22}^{(2)} = \begin{array}{c} \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{cc} n_3 & n_4 \end{array} \\ \left[\begin{array}{cc} D^{(2)} & \\ & E^{(2)} \end{array} \right] \end{array}$$

where the diagonal entries of $D^{(2)} = \mathrm{diag}(d_{ii}^{(2)})$ are in absolute-value-descending order and the diagonal elements of $E^{(2)}$ are smaller than $\varepsilon |d_{11}^{(2)}|$

2. *Early Exit:* If $n_3 = 0$, then $A_{22}^{(1)} = 0$ and by setting $E^{(2)} = 0$, we have

$$A^{(1)} = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left[\begin{array}{cc} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{12}^{(1)T} & 0 \end{array} \right] \end{array} \quad \text{and} \quad B^{(1)} = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left[\begin{array}{cc} I & \\ & 0 \end{array} \right] \end{array},$$

Then

- if $n_1 < n_2$, $A - \lambda B$ is singular. Program exits with output parameter `(K(1),K(2)) = (-1, 2)`.
- if $n_1 \geq n_2$, we reveal the rank of $A_{12}^{(1)}$ by QR decomposition with pivoting:

$$A_{12}^{(1)} P_{12}^{(2)} = Q_{12}^{(2)} \left[\begin{array}{c} A_{13}^{(2)} \\ 0 \end{array} \right]$$

  where the diagonal entries in $A_{13}^{(2)}$ are ordered in absolute-value-descending order.

  (a) If $n_1 = n_2$ and $A_{12}^{(1)}$ is rank deficient, then $A - \lambda B$ is singular. Program exits with output parameter `(K(1),K(2)) = (-1, 3)`.

  (b) If $n_1 = n_2$ and $A_{12}^{(1)}$ is full rank, then $A - \lambda B$ is regular, but no finite eigenvalues. Program exits with output parameter `(K(1),K(2)) = (0, 2)`.

  (c) If $n_1 > n_2$ and $A_{12}^{(1)}$ is rank deficient, then $A - \lambda B$ is singular. Program exits with output parameter `(K(1),K(2)) = (-1, 4)`.

  (d) If $n_1 > n_2$ and $A_{12}^{(1)}$ is full column rank, then there are $n_1 - n_2$ ε-stable eigenvalues, which are the eigenvalues of

$$A^{(2)} U = B^{(2)} U \Lambda \tag{3}$$

  where

$$A^{(2)} = Q_2^T A^{(1)} Q_2 = \begin{array}{c} \\ n_2 \\ n_1 - n_2 \\ n_2 \end{array} \begin{array}{c} \begin{array}{ccc} n_2 & n_1 - n_2 & n_2 \end{array} \\ \left[\begin{array}{ccc} A_{11}^{(2)} & A_{12}^{(2)} & A_{13}^{(2)} \\ A_{12}^{(2)T} & A_{22}^{(2)} & \\ A_{13}^{(2)T} & & 0 \end{array} \right] \end{array},$$

$$B^{(2)} = Q_2^T B^{(1)} Q_2 = \begin{array}{c} \\ n_2 \\ n_1 - n_2 \\ n_2 \end{array} \begin{array}{c} \begin{array}{ccc} n_2 & n_1 - n_2 & n_2 \end{array} \\ \left[\begin{array}{ccc} I & & \\ & I & \\ & & 0 \end{array} \right] \end{array}$$

  and

$$Q_2 = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left[\begin{array}{cc} Q_{12}^{(2)} & \\ & P_{12}^{(2)} \end{array} \right] \end{array}.$$

Let

$$U = \begin{array}{c} \scriptstyle n_2 \\ \scriptstyle n_1-n_2 \\ \scriptstyle n_2 \end{array} \overset{\scriptstyle n_1-n_2}{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}}$$

Then the eigenvalue problem (3) are solved by

$$\begin{aligned} U_1 &= 0 \\ A_{22}^{(2)} U_2 &= U_2 \Lambda \\ U_3 &= -(A_{13}^{(2)})^{-1} A_{12}^{(2)} U_2 \end{aligned}$$

Consequently, $n_1 - n_2$ ε-stable eigenpairs of the original GSEP (1) are $(\Lambda, X = Q_1 R_1 Q_2 U)$. Program exits with output parameter `(K(1),K(2)) = (`$n_1 - n_2$`, 2)`.

3. Set $E^{(2)} = 0$, and update $A^{(1)}$ and $B^{(1)}$:

$$A^{(2)} = Q_2^T A^{(1)} Q_2, \quad B^{(2)} = Q_2^T B^{(1)} Q_2$$

where

$$Q_2 = \begin{array}{c} \scriptstyle n_1 \\ \scriptstyle n_2 \end{array} \overset{\begin{array}{cc} \scriptstyle n_1 & \scriptstyle n_2 \end{array}}{\begin{bmatrix} I & \\ & Q_{22}^{(2)} \end{bmatrix}}$$

4. *Early Exit:* If $n_4 = 0$, then $A_{22}^{(1)}$ is a ε-well-conditioned matrix. We solve the eigenvalue problem

$$A^{(2)} U = B^{(2)} U \Lambda \tag{4}$$

where

$$A^{(2)} = \begin{array}{c} \scriptstyle n_1 \\ \scriptstyle n_2 \end{array} \overset{\begin{array}{cc} \scriptstyle n_1 & \scriptstyle n_2 \end{array}}{\begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ A_{12}^{(2)T} & D^{(2)} \end{bmatrix}} \quad \text{and} \quad B^{(2)} = \begin{array}{c} \scriptstyle n_1 \\ \scriptstyle n_2 \end{array} \overset{\begin{array}{cc} \scriptstyle n_1 & \scriptstyle n_2 \end{array}}{\begin{bmatrix} I & \\ & 0 \end{bmatrix}}$$

Let

$$U = \begin{array}{c} \scriptstyle n_1 \\ \scriptstyle n_2 \end{array} \overset{\scriptstyle n_1}{\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}}$$

The eigenvalue problem (4) becomes

$$\begin{aligned} (A_{11}^{(2)} - A_{12}^{(2)} (D^{(2)})^{-1} A_{12}^{(2)T}) U_1 &= U_1 \Lambda \\ U_2 &= -(D^{(2)})^{-1} (A_{12}^{(2)})^T U_1 \end{aligned}$$

Consequently, n_1 ε-stable eigenpairs of the original GSEP (1) are $(\Lambda, X = Q_1 R_1 Q_2 U)$. Program exits with output parameter `(K(1),K(2)) = (`n_1`, 3)`.

- **Phase 3.**

1. If $n_4 \neq 0$, then $A_{22}^{(1)}$ is ε-ill-conditioned. $A^{(2)}$ and $B^{(2)}$ can be written as 3 by 3 blocks:

$$A^{(2)} = \begin{array}{c} \\ n_1 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_3 & n_4 \end{array} \\ \left[\begin{array}{ccc} A_{11}^{(2)} & A_{12}^{(2)} & A_{13}^{(2)} \\ A_{12}^{(2)T} & D^{(2)} & \\ A_{13}^{(2)T} & & 0 \end{array} \right] \end{array} \quad \text{and} \quad B^{(2)} = \begin{array}{c} \\ n_1 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_3 & n_4 \end{array} \\ \left[\begin{array}{ccc} I & & \\ & 0 & \\ & & 0 \end{array} \right] \end{array}$$

where $n_3 + n_4 = n_2$.

2. *Early Exit*: If $n_1 < n_4$, then $A - \lambda B$ is singular. Program exits with output parameter `(K(1),K(2)) = (-1, 5)`.

3. When $n_1 \geq n_4$, we reveal the rank of $A_{13}^{(2)}$ by QR decomposition with pivoting:

$$A_{13}^{(2)} P_{13}^{(3)} = Q_{13}^{(3)} R_{13}^{(3)}$$

where

$$R_{13}^{(3)} = \begin{array}{c} \\ n_4 \\ n_5 \end{array} \begin{array}{c} n_4 \\ \left[\begin{array}{c} A_{14}^{(3)} \\ 0 \end{array} \right] \end{array}$$

4. *Early Exit:* (a) If $n_1 = n_4$ and $A_{13}^{(2)}$ is rank deficient, then $A - \lambda B$ is singular. Program exits with output parameter `(K(1),K(2)) = (-1, 6)`.

   (b) If $n_1 = n_4$ and $A_{13}^{(2)}$ is full rank, then $A - \lambda B$ is regular, but no finite eigenvalues. Program exits with output parameter `(K(1),K(2)) = (0, 3)`.

   (c) If $n_1 > n_4$ and $A_{13}^{(2)}$ is rank deficient, $A - \lambda B$ is singular. Program exits with output parameter `(K(1),K(2)) = (-1, 7)`.

5. Update

$$A^{(3)} = Q_3^T A^{(2)} Q_3 \quad \text{and} \quad B^{(3)} = Q_3^T B^{(2)} Q_3$$

where

$$Q_3 = \begin{array}{c} \\ n_1 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_3 & n_4 \end{array} \\ \left[\begin{array}{ccc} Q_{13}^{(3)} & & \\ & I & \\ & & P_{13}^{(3)} \end{array} \right] \end{array}$$

6. By the rank-revealing decomposition, matrices $A^{(3)}$ and $B^{(3)}$ can be written as 4×4 blocks:

$$A^{(3)} = \begin{array}{c} \\ n_4 \\ n_5 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{cccc} n_4 & n_5 & n_3 & n_4 \end{array} \\ \left[\begin{array}{cccc} A_{11}^{(3)} & A_{12}^{(3)} & A_{13}^{(3)} & A_{14}^{(3)} \\ (A_{12}^{(3)})^T & A_{22}^{(3)} & A_{23}^{(3)} & 0 \\ (A_{13}^{(3)})^T & (A_{23}^{(3)})^T & D^{(2)} & 0 \\ (A_{14}^{(3)})^T & 0 & 0 & 0 \end{array} \right] \end{array} \quad \text{and} \quad B^{(3)} = \begin{array}{c} \\ n_4 \\ n_5 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{cccc} n_4 & n_5 & n_3 & n_4 \end{array} \\ \left[\begin{array}{cccc} I & & & \\ & I & & \\ & & 0 & \\ & & & 0 \end{array} \right] \end{array},$$

   where $n_1 = n_4 + n_5$ and $n_2 = n_3 + n_4$. The ε-stable eigenpairs of the GSEP (1) are given by the finite eigenvalues of

$$A^{(3)} U = B^{(3)} U \Lambda \tag{5}$$

Let

$$U = \begin{array}{c} \\ n_4 \\ n_5 \\ n_3 \\ n_4 \end{array} \begin{array}{c} n_5 \\ \left[\begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array}\right] \end{array}$$

then the eigenvalue problem (5) is equlvalent to the following expressions:

$$\begin{aligned} U_1 &= 0 \\ \left(A_{22}^{(3)} - A_{23}^{(3)}(D^{(3)})^{-1}A_{23}^{(3)T}\right) U_2 &= U_2\Lambda \\ U_3 &= -(D^{(2)})^{-1}A_{23}^{(3)T}U_2 \\ U_4 &= -(A_{14}^{(3)})^{-1}\left(A_{12}^{(3)}U_2 + A_{13}^{(3)}U_3\right) \end{aligned}$$

Consequently, n_5 ε-stable eigenpairs of the GSEP (1) are given by $(\Lambda, X = Q_1R_1Q_2Q_3U)$. Program exits with output parameter `(K(1),K(2)) = (`n_5`, 4)`.

4. Numerical examples. We design five test cases to illustrate major features of the routine `DSYGVIC`. For all these cases,

$$A = Q^T HQ \quad \text{and} \quad B = Q^T SQ$$

where Q is a random orthogonal matrix, and H and S are prescribed to be of certain structure for testing the different cases of the algorithm. Similar to the test of LAPACK routine `DSYGV`, the correctness of the routine `DSYGVIC` is measured by the following two residuals for computed eigenpairs $(\widehat{X}, \widehat{\Lambda})$:

$$\text{Res1} = \frac{\|A\widehat{X} - B\widehat{X}\widehat{\Lambda}\|}{\|A\|\,\|\widehat{X}\| + \|B\|\,\|\widehat{X}\|\,\|\widehat{\Lambda}\|} \quad \text{and} \quad \text{Res2} = \frac{\|\widehat{X}^T B\widehat{X} - I\|}{\|B\|\,\|\widehat{X}\|}$$

1. **Test case 1.** Consider 10×10 matrices $A = Q^T HQ$ and $B = Q^T SQ$, where

$$H = \left[\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

and

$$S = \text{diag}[1, 2, 3, 2, 1, 1, 2, 3, 1, 2]$$

This is the case where B is positive definite and well-conditioned.

LAPACK routine `DSYGV` returns 10 eigenvalues with `INFO` $= 0$. New routine `DSYGVIC` with $\varepsilon = 10^{-12}$ also returns 10 eigenvalues with `INFO` $= 0$. The computed eigenvalues agree to machine precision, with the comparable accuracy as shown in the following table:

	INFO	#eigvals	Res1	Res2
DSYGV	0	10	5.48e-17	2.41e-16
DSYGVIC	0	10	7.32e-17	2.38e-16

The output parameter (K(1),K(2))=(10,1) of DSYGVIC indicates that the matrix B is well-conditioned, and there are full set of finite eigenvalues of (A, B). The original GSEP is reduced to the eigenvalue problem (2).

2. **Test case 2**. Consider 8×8 matrices $A = Q^T HQ$ and $B = Q^T SQ$, where

$$H = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$S = \text{diag}[1, 1, 1, 1, \delta, \delta, \delta, \delta]$$

This is same test case used by Fix and Heiberger [2]. It is known that as $\delta \to 0$, $\lambda = 3, 4$ are the only stable eigenvalues.

Consider $\delta = 10^{-15}$, the following table shows the computed eigenvalues by LAPACK routine DSYGV and new routine DSYGVIC with the threshold $\varepsilon = 10^{-12}$.

λ_i	DSYGV	DSYGVIC
1	-0.3229260685047438e+08	**0.3000000000000001e+01**
2	-0.3107213627119420e+08	**0.3999999999999999e+01**
3	**0.2957918878610765e+01**	
4	**0.4150528124449937e+01**	
5	0.3107214204558684e+08	
6	0.3229261357421688e+08	
7	0.1004773743630529e+16	
8	0.2202090698823234e+16	

As we can see DSYGV returns all 8 eigenvalues including 6 unstable ones. For the two stable eigenvalues, there is significant loss of accuracy. In contrast, DSYGVIC only returns two stable eigenvalues to full machine precision.

3. **Test case 3**. Consider 10×10 matrices $A = Q^T HQ$ and $B = Q^T SQ$, where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$S = \text{diag}[1, 2, 3, 2, 1, 1, 2\delta, 3\delta, \delta, 2\delta]$$

Note that B is very ill-conditioned for small δ. Furthermore, the matrix H is designed such that the reduced matrix pair is of the form (3) with $n_1 = 6$, $n_2 = 4$ and $n_3 = 0$.

Consider $\delta = 10^{-15}$, LAPACK routine `DSYGV` treats B as a positive definite matrix and runs successfully with `INFO` $= 0$, but with significant loss of accuracy as shown in the following table. But `DSYGVIC` with the threshold $\varepsilon = 10^{-12}$ computes two stable eigenvalues to machine precision.

	`INFO`	#eigvals	Res1	Res2
`DSYGV`	0	10	9.72e-11	5.08e-10
`DSYGVIC`	0	2	1.04e-16	8.20e-17

If $\delta = 10^{-17}$, LAPACK routine `DSYGV` detects B is not positive definite, and returns immediately with `INFO` $= 17$. In contrast, the new routine `DSYGVIC` with the threshold $\varepsilon = 10^{-12}$ successfully completes the computation and reports there are two ε-stable eigenvalues with full machine accuracy:

	`INFO`	#eigvals	Res1	Res2
`DSYGV`	17	–	–	–
`DSYGVIC`	0	2	1.01e-16	1.12e-16

The output parameter `(K(1),K(2))=(2,2)` of `DSYGVIC` indicates that the program exits at the case that returns $n_1 - n_2$ eigenvalues.

4. **Test case 4**. Consider 10×10 matrices $A = Q^T H Q$ and $B = Q^T S Q$, where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$S = \text{diag}[1, 2, 3, 2, 1, 1, 2\delta, 3\delta, \delta, 2\delta],$$

where matrices H and S are designed such that the reduced eigenvalue problem is of the form (4) with $n_1 = 6$, $n_2 = 4$ and $n_4 = 0$ as B becomes ill-conditioned.

Consider $\delta = 10^{-15}$, LAPACK routine `DSYGV` treats B as a positive definite matrix and runs successfully with `INFO` $= 0$, but with significant loss of accuracy as shown in the following table. But `DSYGVIC` with the threshold $\varepsilon = 10^{-12}$ computes six stable eigenvalues to machine precision.

	`INFO`	#eigvals	Res1	Res2
`DSYGV`	0	10	5.50e-3	5.58e-10
`DSYGVIC`	0	6	2.45e-16	9.72e-16

If $\delta = 10^{-17}$, LAPACK routine `DSYGV` detects B is not positive definite, and returns immediately with `INFO` $= 17$. In contrast, the new routine `DSYGVIC` with $\varepsilon = 10^{-12}$ returns 6 ε-stable eigenvalues with the accuracy

	`INFO`	#eigvals	Res1	Res2
`DSYGV`	17	–	–	–
`DSYGVIC`	0	6	8.30e-17	2.02e-16

The output parameter `(K(1),K(2))=(6,3)` of `DSYGVIC` indicates that the program exits at the case that returns n_1 eigenvalues.

5. **Test case 5**. Consider 10×10 matrices $A = Q^T H Q$ and $B = Q^T S Q$, where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$S = \mathrm{diag}[1, 2, 3, 2, 1, 1, 2\delta, 3\delta, \delta, 2\delta],$$

where H and S are designed such that the reduced eigenvalue problem is of the form (5) with $n_1 = 6, n_2 = 4, n_3 = 2, n_4 = 2$ and $n_5 = 4$ as $\delta \to 0$.

Consider $\delta = 10^{-17}$, LAPACK routine `DSYGV` detects B is not positive definite, and returns immediately with `INFO` $= 17$. In contrast, the new routine `DSYGVIC` with $\varepsilon = 10^{-12}$ returns 4 ε-stable eigenvalues with the accuracy

	`INFO`	#eigvals	Res1	Res2
`DSYGV`	17	–	–	–
`DSYGVIC`	0	4	8.49e-17	1.95e-16

The output parameter `(K(1),K(2))=(4,4)` of `DSYGVIC` indicates that the program exits at the case that returns n_5 eigenvalues.

5. To do.

- CPU timing benchmark for large size n.
- Applications
- ...

References

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